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# Estimating the Somos' quadratic recurrence constant

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ABSTRACT

*Text.* The aim of this paper is to provide some estimates about the Somos' quadratic recurrence constant, using its relation with the generalized Euler constant.

*Video.* For a video summary of this paper, please click here or visit http://www.youtube.com/watch?v=3QjmHit3mC4.



## 1. Introduction

In 1999 Somos [7] defined the sequence $g_n = ng_{n-1}^2$, with $g_0 = 1$, then Finch [1, p. 446] proved the asymptotic formula

$$g_n = \sigma^{2^n}\left(n + 2 - \frac{1}{n} + \frac{4}{n^2} - \frac{21}{n^3} + \frac{138}{n^4} - \frac{1091}{n^5} + \cdots\right)^{-1} \quad (n \to \infty).$$

Here the constant $\sigma = 1.661687949\ldots$ is now known as the Somos' quadratic recurrence constant. As this constant appears in important problems from pure and applied analysis, it attracted in the recent past the attention of many authors. Defining expressions for $\sigma$ include the formulas

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$$\begin{aligned}\sigma &= \sqrt{1\sqrt{2\sqrt{3\sqrt{4\cdots}}}}\\ &= \prod_{k=1}^{\infty} k^{1/2^k}\\ &= \prod_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^{1/2^k}\\ &= \exp\left\{\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k \ln k\right\},\end{aligned}$$

or integral representations

$$\begin{aligned}\sigma &= \exp\left\{-\int_0^1 \frac{1-x}{(2-x)\ln x}\,dx\right\}\\ &= \exp\left\{-\int_0^1\int_0^1 \frac{x}{(2-xy)\ln(xy)}\,dx\,dy\right\}.\end{aligned}$$

See [2,5,6].

Very recently, Sondow and Hadjicostas [8] and Pilehrood and Pilehrood [4] introduced the function $\gamma^*(z) = z\gamma(z)$, where

$$\gamma(z) = \sum_{k=1}^{\infty} z^{k-1}\left(\frac{1}{k} - \ln\frac{k+1}{k}\right) \quad \bigl(|z| \leqslant 1\bigr)$$

is the generalized Euler-constant function ($\gamma(1) = 0.577215\ldots$ is the classical Euler-constant).

An alternative method in estimating the generalized Euler-constant function $\gamma(z)$ was proposed by Lampret [3] who exploited the Euler–Maclaurin (Boole/Hermite) summation formula.

These functions are closely related with the Somos' quadratic recurrence constant $\sigma$, since

$$\gamma\left(\frac{1}{2}\right) = 2\ln\frac{2}{\sigma},$$

or more precisely,

$$\sigma = 2\exp\left\{-\frac{1}{2}\gamma\left(\frac{1}{2}\right)\right\}. \tag{1.1}$$

## 2. Estimating $\gamma(1/2)$

In order to deduce some estimates for the $\sigma$ constant, we evaluate the series

$$\gamma\left(\frac{1}{2}\right) = \sum_{k=1}^{\infty}\frac{1}{2^{k-1}}\left(\frac{1}{k} - \ln\frac{k+1}{k}\right).$$

First we need the following intermediary result.

**Lemma 1.** *For every integer positive $k$, we define*

$$a(k)=\frac{1}{k^2+\frac{8}{3}k-\frac{79}{18}+\frac{3244}{135k}},\qquad b(k)=\frac{1}{k^2+\frac{8}{3}k-\frac{79}{18}}.$$

*Then for every integer $k\geqslant 2$, we have*

$$a(k)-\frac{1}{2}a(k+1)<\frac{1}{k}-\ln\frac{k+1}{k}<b(k)-\frac{1}{2}b(k+1). \tag{2.1}$$

**Proof.** It suffices to show that $f(x)<0$ and $g(x)>0$ for all $x>1$, where

$$f(x)=a(x)-\frac{1}{2}a(x+1)-\left(\frac{1}{x}-\ln\frac{x+1}{x}\right)$$

and

$$g(x)=b(x)-\frac{1}{2}b(x+1)-\left(\frac{1}{x}-\ln\frac{x+1}{x}\right).$$

We have

$$f'(x)=\frac{P(x)}{x^2(x+1)(720x^2-1185x+270x^3+6488)^2(1065x+1530x^2+270x^3+6293)^2};$$
$$g'(x)=-\frac{Q(x)}{x^2(x+1)(48x+18x^2-79)^2(84x+18x^2-13)^2},$$

where

$$\begin{aligned}P(x)={}&1117\,487\,310\,968\,745x^2-44\,706\,512\,610\,480x+273\,293\,662\,407\,930x^3\\&+337\,902\,921\,739\,125x^4+143\,943\,233\,452\,200x^5+46\,243\,652\,376\,600x^6\\&+21\,593\,578\,509\,000x^7+2987\,475\,898\,500x^8+1667\,005\,934\,472\,256\end{aligned}$$

and

$$Q(x)=14\,912\,040x-62\,160\,084x^2+26\,980\,992x^3+37\,264\,860x^4+6306\,336x^5-1054\,729.$$

As $P(x+1)$ and $Q(x+1)$ have all coefficients positive, it results that $f$ is strictly increasing and $g$ is strictly decreasing. But $f(\infty)=g(\infty)=0$, so $f<0$ and $g>0$ on $[1,\infty)$ and the conclusion follows. □

By adding inequalities of the form

$$\frac{a(k)}{2^{k-1}}-\frac{a(k+1)}{2^k}<\frac{1}{2^{k-1}}\left(\frac{1}{k}-\ln\frac{k+1}{k}\right)<\frac{b(k)}{2^{k-1}}-\frac{b(k+1)}{2^k}$$

from $k = n+1$ to $k = \infty$, we get

$$\frac{a(n+1)}{2^n} < \sum_{k=n+1}^{\infty} \frac{1}{2^{k-1}}\left(\frac{1}{k} - \ln\frac{k+1}{k}\right) < \frac{b(n+1)}{2^n}. \tag{2.2}$$

This double inequality gives the error estimate when $\gamma(\frac{1}{2})$ is approximated by

$$\gamma_n\left(\frac{1}{2}\right) = \sum_{k=1}^{n} \frac{1}{2^{k-1}}\left(\frac{1}{k} - \ln\frac{k+1}{k}\right),$$

so we can state the following

**Theorem 1.** *For every $n \geqslant 1$, we have*

$$\gamma_n\left(\frac{1}{2}\right) + \frac{270(n+1)}{2^n(1065n + 1530n^2 + 270n^3 + 6293)} < \gamma\left(\frac{1}{2}\right) < \gamma_n\left(\frac{1}{2}\right) + \frac{18}{2^n(84n + 18n^2 - 13)}. \tag{2.3}$$

**Proof.** The double inequality (2.2) can be equivalently written as

$$\frac{a(n+1)}{2^n} < \gamma\left(\frac{1}{2}\right) - \gamma_n\left(\frac{1}{2}\right) < \frac{b(n+1)}{2^n}$$

and the conclusion follows if we take into account that

$$\frac{a(n+1)}{2^n} = \frac{270(n+1)}{2^n(1065n + 1530n^2 + 270n^3 + 6293)}$$

and

$$\frac{b(n+1)}{2^n} = \frac{18}{2^n(84n + 18n^2 - 13)}. \qquad \Box$$

From (2.3) we can prove the following result which has a simpler form than (2.3), although it is slightly weaker than (2.3).

**Corollary 1.** *For every integer $n \geqslant 8$, we have*

$$\gamma_n\left(\frac{1}{2}\right) + \frac{1}{(n+3)^2 2^n} < \gamma\left(\frac{1}{2}\right) < \gamma_n\left(\frac{1}{2}\right) + \frac{1}{(n+2)^2 2^n}.$$

This follows by Theorem 1 and taking into account that

$$\begin{aligned}&\frac{270(n+1)}{2^n(1065n + 1530n^2 + 270n^3 + 6293)} - \frac{1}{(n+3)^2 2^n}\\ &\quad = \frac{2985n + 360n^2 - 3863}{2^n(n+3)^2(1065n + 1530n^2 + 270n^3 + 6293)} > 0\end{aligned}$$

and

$$\frac{1}{(n+2)^2 2^n} - \frac{18}{2^n(84n+18n^2-13)} = \frac{12n-85}{2^n(n+2)^2(84n+18n^2-13)} > 0.$$

Finally, using (1.1), we obtain the following estimates for the $\sigma$ constant.

**Corollary 2.** *For every integer $n \geqslant 8$, we have*

$$2\exp\left\{-\frac{1}{2}\gamma_n\left(\frac{1}{2}\right) - \frac{1}{(n+2)^2 2^{n+1}}\right\} < \sigma < 2\exp\left\{-\frac{1}{2}\gamma_n\left(\frac{1}{2}\right) - \frac{1}{(n+3)^2 2^{n+1}}\right\}.$$

## 3. Further estimates and concluding remarks

Sondow and Hadjicostas [8] defined the generalized Somos constant

$$\sigma_t = \sqrt[t]{1\sqrt[t]{2\sqrt[t]{3\sqrt[t]{4\cdots}}}} = \left(\frac{t}{t-1}\right)^{1/(t-1)} \exp\left\{-\frac{1}{t(t-1)}\gamma\left(\frac{1}{t}\right)\right\}, \tag{3.1}$$

where $t > 1$. Our above method can be adapted to estimate $\sigma_t$ in terms of the polylogarithm function

$$Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

(this definition is valid for all complex numbers $s$ and $z$ where $|z| < 1$).

In order to estimate $\sigma_3$, we give the following intermediary result.

**Lemma 2.** *For every integer $k$, we define*

$$c(k) = \frac{-1}{2k^3 + \frac{9}{2}k^2 - \frac{243}{40}k + \frac{3821}{160} - \frac{3022137}{22400k}}, \qquad d(k) = \frac{-1}{2k^3 + \frac{9}{2}k^2 - \frac{243}{40}k + \frac{3821}{160}}.$$

*Then for every $k \geqslant 3$, we have*

$$c(k) - \frac{1}{3}c(k+1) < \left(\frac{1}{k} - \ln\frac{k+1}{k}\right) - \frac{1}{2k^2} < d(k) - \frac{1}{3}d(k+1). \tag{3.2}$$

**Proof.** It suffices to show that $u < 0$ and $v > 0$ on $[3, \infty)$, where

$$u(x) = c(x) - \frac{1}{3}c(x+1) - \left[\left(\frac{1}{x} - \ln\frac{x+1}{x}\right) - \frac{1}{2x^2}\right]$$

and

$$v(x) = d(x) - \frac{1}{3}d(x+1) - \left[\left(\frac{1}{x} - \ln\frac{x+1}{x}\right) - \frac{1}{2x^2}\right].$$

For every $x \geqslant 3$, we have

$$u'(x) = R(x)\big[x^3(x+1)\big(44800x^4 + 280000x^3 + 435120x^2 + 744380x - 2477677\big)^2 \\ \times \big(44800x^4 + 100800x^3 - 136080x^2 + 534940x - 3022137\big)^2\big]^{-1},$$

where $R(x+2)$ is a polynomial of degree 11 with positive coefficients, and

$$v'(x) = -S(x)\big[x^3(x+1)\big(320x^3 + 720x^2 - 972x + 3821\big)^2 \\ \times \big(320x^3 + 1680x^2 + 1428x + 3889\big)^2\big]^{-1},$$

where $S(x)$ is a polynomial of degree 8 with positive coefficients. It follows that $u$ is strictly increasing in $[3,\infty)$, while $v$ is strictly decreasing in $[3,\infty)$. Since also $u(\infty) = v(\infty) = 0$, we conclude that $u(x) < 0$ and $v(x) > 0$ for $x \geqslant 3$. $\square$

By adding inequalities of the form

$$\frac{c(k)}{3^{k-1}} - \frac{c(k+1)}{3^k} < \frac{1}{3^{k-1}}\left(\frac{1}{k} - \ln\frac{k+1}{k}\right) - \frac{1}{2(3^{k-1})k^2} < \frac{d(k)}{3^{k-1}} - \frac{d(k+1)}{3^k}$$

from $k = n+1$ to $k = \infty$, we get

$$\frac{c(n+1)}{3^n} < \sum_{k=n+1}^{\infty} \frac{1}{3^{k-1}}\left(\frac{1}{k} - \ln\frac{k+1}{k}\right) - \sum_{k=n+1}^{\infty} \frac{1}{2(3^{k-1})k^2} < \frac{d(n+1)}{3^n}.$$

By designating

$$\gamma_n(z) = \sum_{k=1}^{n} z^{k-1}\left(\frac{1}{k} - \ln\frac{k+1}{k}\right), \quad |z| < 1,$$

and taking into account that

$$c(n+1) = -\frac{22400(n+1)}{44800n^4 + 280000n^3 + 435120n^2 + 744380n - 2477677},$$
$$d(n+1) = -\frac{160}{320n^3 + 1680n^2 + 1428n + 3889},$$

we are in a position to give the following

**Theorem 2.** *For every $n \geqslant 3$, we have*

$$h_n < \gamma\left(\frac{1}{3}\right) < j_n,$$

*where*

$$h_n = \gamma_n\left(\frac{1}{3}\right) - \frac{22400(n+1)}{(44800n^4+280000n^3+435120n^2+744380n-2477677)3^n} + \frac{3}{2}Li_2\left(\frac{1}{3}\right) - \frac{3}{2}\sum_{k=1}^{n}\frac{1}{3^k k^2},$$

$$j_n = \gamma_n\left(\frac{1}{3}\right) - \frac{160}{(320n^3+1680n^2+1428n+3889)3^n} + \frac{3}{2}Li_2\left(\frac{1}{3}\right) - \frac{3}{2}\sum_{k=1}^{n}\frac{1}{3^k k^2}.$$

Using (3.1), we can easily state the following

**Corollary 3.** *For every integer $n \geqslant 3$, we have*

$$\left(\frac{3}{2}\right)^{1/2}\exp\left(-\frac{j_n}{6}\right) < \sigma_3 < \left(\frac{3}{2}\right)^{1/2}\exp\left(-\frac{h_n}{6}\right).$$

By using the bounds

$$\frac{160}{320n^3+1680n^2+1428n+3889} > \frac{1}{2(n+\frac{7}{4})^3},$$

$$\frac{22400(n+1)}{(44800n^4+280000n^3+435120n^2+744380n-2477677)} < \frac{1}{2(n+\frac{6}{4})^3},$$

the telescoping inequalities

$$\frac{1}{3^{n-1}(3n^2+n)} - \frac{1}{3^n(3(n+1)^2+(n+1))} < \frac{1}{3^n n^2} < \frac{1}{3^{n-1}(2n^2+n)} - \frac{1}{3^n(2(n+1)^2+(n+1))}$$

and Theorem 2, we obtain the following estimates of simpler form.

**Corollary 4.** *For every integer $n \geqslant 3$, we have*

$$\gamma_n\left(\frac{1}{3}\right) - \frac{1}{2(n+\frac{6}{4})^3} + \frac{1}{2(n+1)(3n+4)3^{n-1}} < \gamma\left(\frac{1}{3}\right) < \gamma_n\left(\frac{1}{3}\right) - \frac{1}{2(n+\frac{7}{4})^3} + \frac{1}{2(n+1)(2n+3)3^{n-1}}.$$

Finally, an idea to estimate $\sigma_t$ for $t \geqslant 4$, is to find two functions of the form

$$\alpha(x) = \frac{1}{\sum_{k=-1}^{t} w_k x^k}, \qquad \beta(x) = \frac{1}{\sum_{k=0}^{t} w_k x^k},$$

where $w_{-1}, w_0, \ldots, w_t \in \mathbb{R}$ such that the functions $\alpha(x) - \alpha(x+1)/t$ and $\beta(x) - \beta(x+1)/t$ have expansions of the form $\sum_{i=t}^{\infty} \frac{r_i}{x^i}$, that match as many coefficients of the expansion of

$$\left(\frac{1}{x}-\ln\frac{x+1}{x}\right)-\sum_{i=2}^{t-1}\frac{(-1)^i}{ix^i}=\sum_{j=t}^{\infty}\frac{(-1)^j}{jx^j}$$

as possible. The existence and construction of such general functions satisfying

$$\alpha(x)-\frac{1}{t}\alpha(x+1)<\left(\frac{1}{x}-\ln\frac{x+1}{x}\right)-\sum_{i=2}^{t-1}\frac{(-1)^i}{ix^i}<\beta(x)-\frac{1}{t}\beta(x+1)$$

remains an open problem which is possible to be solved by recursion arguments, or using a completely different method.

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## Supplementary material

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